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On Schur algebras, Ringel duality and symmetric groups

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Abstract

In Erdmann and Henke (Math. Proc. Cambridge Philos. Soc., to appear) we determine precisely the degrees r for which the Schur algebra $S(2,r)$ is its own Ringel dual. Here we study some applications: We classify uniserial Weyl modules and tilting modules. Based on Doty (J. Algebra 95 (1985) 373), we describe the submodule lattice of Specht modules labelled by two-part partitions and we classify uniserial Specht modules and Young modules labelled by two-part partitions. Moreover, we determine extensions for simple modules for the Ringel duals of arbitrary $S(2,r)$. As a consequence we obtain corresponding results on symmetric groups.

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0. Introduction

Schur algebras are an important class of quasi-hereditary algebras; the module category of the Schur algebra $S(n,r)$ over an infinite field is equivalent to the category of r-homogeneous polynomial representations of $GL_n(K)$; and its Ringel dual is closely related to the group algebra of the symmetric group $\mathscr{S}_r$. In [10] we determined the degrees r for which the Schur algebra $S(2,r)$ is its own Ringel dual. Here we study some consequences, in particular we apply these results to the representation theory of symmetric groups.

Let E be a two-dimensional vector space over K, then the r-fold tensor product $E^{\otimes r}$ is a permutation module for the symmetric group $\mathscr{S}_r$, and the Schur algebra $S(2,r)$

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can be defined as the endomorphism ring,

$$S(2,r) = \mathrm{End}_{\mathscr{S}_r}(E^{\otimes r}).$$

Let $I_r \subset K\mathscr{S}_r$ be the kernel of the action of $K\mathscr{S}_r$ on the tensor space $E^{\otimes r}$, then $\mathrm{End}_{S(2,r)}(E^{\otimes r}) \cong K\mathscr{S}_r/I_r =: \overline{K\mathscr{S}_r}$, and moreover, this is also a Ringel dual $S(2,r)'$ for $S(2,r)$, except for a small modification if $p=2$ and r is even, see Section 1.2.

We proved in [10] that $S(2,r)'$ is Morita equivalent to $S(2,r)$ if and only if $r \leqslant p^2$, or r is of the form (ap^k-2) or $(ap^k-2)\pm 1$ where $2 \leqslant a \leqslant p$ and $k \geqslant 1$. We call such a degree r a *self-dual* degree. So if r is a self-dual degree then the factor algebra $\overline{K\mathscr{S}_r}$ is Morita equivalent to the Schur algebra $S(2,r)$ (with the modification if $p=2$ and r is even). If d is arbitrary then there is some self-dual degree $r \geqslant d$ with $r \equiv d \bmod 2$. Then $S(2,d)'$, and hence $\overline{K\mathscr{S}_d}$, is Morita equivalent to an algebra $eS(2,r)e$ where e is a *good* idempotent (see Section 1.1).

One consequence is the following. The Weyl modules of $GL_d(K)$ corresponding to partitions with at most two columns can be identified with the standard modules for the Ringel dual of $S(d,d)$, by Donkin [5]. These are the same as the standard modules for $S(2,d)'$; so they are identifiable with modules $e\Delta(\lambda)$ where e is as above and λ is a partition of r with at most two parts. By Doty [7] one can describe the submodule lattice of such modules. The Weyl modules corresponding to partitions with at most two columns were determined by Adamovich (in Russian [1]), in [16] the results are summarized.

We are interested in the submodule structure of Specht modules and Young modules of symmetric groups. For Schur algebras $S(2,r)$, we classify precisely which (standard) Weyl modules and which tilting modules are uniserial, for arbitrary degree r. This gives the classification of uniserial Specht modules and Young modules labelled by two-part partitions. Moreover using [7], we obtain a description of the submodule lattice of Specht modules labelled by two-part partitions.

We determine the quiver of a *good* subalgebra eSe of a general Schur algebra $S{:=}S(2,r)$. That is, we determine $\mathrm{Ext}^1_{eSe}(eL(\lambda), eL(\mu))$ for arbitrary $\lambda,\mu \in \Lambda^+(2,r)$ where e is a *good* idempotent of S (see Section 1.1). As a consequence we obtain $\mathrm{Ext}^1_{\overline{K\mathscr{S}_d}}(D^\lambda, D^\mu)$ for arbitrary two-part partitions λ,μ of d. If $p \geqslant 5$ this is the same as $\mathrm{Ext}^1_{K\mathscr{S}_r}(D^\lambda, D^\mu)$, which was determined in [16] (see however the correction). For $p=2$, parts of the quiver for two-part partitions were determined in [18]. Our result gives a refinement, namely it classifies the extensions on which the ideal I_r acts trivially.

For notation and background we refer to [10,11,14,17,19].

1. Preliminaries

1.1. Let A be a finite-dimensional algebra, with simple modules $L(\lambda)$, $\lambda \in \Lambda$, which is quasi-hereditary with respect to the partial order $(\Lambda, \leqslant)$. We call an idempotent e of A a *good* idempotent if $\Gamma = \{\lambda \in \Lambda\colon eL(\lambda) \neq 0\}$ is a coideal in Λ. If so then the factor algebra $\bar{A}{:=}A/AeA$ is quasi-hereditary, with respect to $\Lambda \setminus \Gamma$ and the same ordering, and where the standard modules and the tilting modules of $\bar{A}$ are the same as those for A, labelled by $\Lambda \setminus \Gamma$. We call such $\bar{A}$ a *good* quotient of A.

Moreover, the algebra eAe is also quasi-hereditary, with respect to Γ, with the same ordering, and with standard modules $e\Delta(\lambda)$ and tilting modules $eT(\lambda)$, for $\lambda \in \Gamma$. We call such an algebra a *good* subalgebra of A. The proof of the following may be found in [13].

Lemma 1.1. (a) *B is a good subalgebra of A if and only if the Ringel dual of B is Morita equivalent to a good quotient of A'.*

(b) *A/I is a good quotient of A if and only if the Ringel dual of A/I is Morita equivalent to a good subalgebra of A'.*

Now let $S = S(2,r)$, then the simple modules are labelled by the set Λ of two-part partitions of r, and the ordering is the dominance order. As explained in [10], instead of $\lambda = (\lambda_1, \lambda_2)$ we use $m = \lambda_1 - \lambda_2$ as a label, if the degree is clear from the context, or is not important. Then $\Lambda = \{m \in \mathbb{N}_0 \mid 0 \leqslant m \leqslant r \text{ and } m \equiv r \bmod 2\}$, with the natural linear order. Hence an idempotent e of S is good precisely if there is some τ such that $eL(m) \neq 0$ if and only if $m \geqslant \tau$.

1.2. It is known that whenever $d < r$ and $d \equiv r \bmod 2$ then $S(2,d)$ is a good quotient of $S(2,r)$ (see [8], or modify the results in [6]). It follows that then $S(2,d)'$ is a good subalgebra of $S(2,r)'$. Consequently, we have that $\overline{K\mathscr{S}_d}$ is a good subalgebra of $\overline{K\mathscr{S}_r}$. If in addition r is a self-dual degree then $\overline{K\mathscr{S}_r}$ is Morita equivalent to $S(2,r)$. The only exception occurs when $p = 2$ and r is even, then $\overline{K\mathscr{S}_r}$ is Morita equivalent to $fS(2,r)'f$ where f is an idempotent with $fL_{S'}(m) = 0$ if and only if $m = 0$. In Section 5.1 we describe how the labellings of the simple modules are related under this equivalence.

Remark. Suppose that $d < r$ and $d \equiv r \bmod 2$. In [12,13] it is proved that there are many such pairs such that $S(2,d)$ is a good subalgebra of $S(2,r)$. These included as special cases self-dual degrees d and r. As we have just seen, $S(2,d)'$ is a good subalgebra of $S(2,r)'$, and hence if both d and r are self-dual then the result of [13] in this case follows, see Section 1.2.

1.3. We will study submodule lattices of modules which are twisted tensor products and we will make use of the following, probably well known result. Let G be any group which satisfies the hypotheses in [15], part II and let $L(i)$ be any simple module which remains simple as a module for the Frobenius kernel G_1 and which has trivial endomorphism ring. Let X be a G-module.

Lemma 1.2. *Let $0 \leqslant i \leqslant p-1$. The functor $(-)^F \otimes L(i)$ induces an isomorphism of submodule lattices of X and $X^F \otimes L(i)$.*

Proof. Let $L=L(i)$. The map $V \to V^F \otimes L$ induces a lattice homomorphism between the submodule lattices of X and of $X^F \otimes L$. Let $M \subset X^F \otimes L$ be a G-submodule, then by Jantzen [15] I, 6.15(2), we have $\mathrm{soc}_{G_1}(M)_L = \mathrm{Hom}_{G_1}(L,M) \otimes L$ as G-modules (identifying $\mathrm{Hom}_{G_1}(L,M)$ with a submodule of X^F). But $\mathrm{soc}_{G_1}(M)_L = M$ and $W := \mathrm{Hom}_{G_1}(L,M)$ is a G-module which is trivial as a G_1-module. So, by Jantzen [15] II.3.16, we have $W = V^F$ for some G-module V, and hence $M = V^F \otimes L$. If $M = V_1^F \otimes L = V_2^F \otimes L$ then by the above $V_1^F = \mathrm{Hom}_{G_1}(L,M) = V_2^F \subseteq X^F$ and then $V_1 = V_2$. □

1.4. We will frequently use the following facts about Weyl modules, see [15,20]. The Weyl modules $\Delta(m)$ are multiplicity-free. We have $\Delta(m)^F \otimes \Delta(p-1) \cong \Delta(mp+p-1)$. Let $0 \leqslant i,j$ with $i+j=p-2$, then there is an exact sequence

$$0 \to \Delta(m-1)^F \otimes L(j) \to \Delta(mp+i) \to \Delta(m)^F \otimes L(i) \to 0.$$

1.5. Recall the following properties of the tilting modules (see [5,10,20]). For $m \leqslant p-1$ one has $T(m)=\Delta(m)=\nabla(m)$. We have $T(wp+p-1) \cong T(w)^F \otimes T(p-1)$. If $m=kp+j$ for $0 \leqslant j \leqslant p-2$ and $k \geqslant 1$ then $T(m) \cong T(p+j) \otimes T(k-1)^F$. Moreover, for any $s \geqslant 0$ there is an exact sequence

$$0 \to \Delta(p(s+1)+j) \to T(p+j) \otimes \Delta(s)^F \to \Delta(ps+i) \to 0.$$

2. Uniserial Weyl modules and tilting modules for $S(2,r)$

In this section we will classify all uniserial Weyl modules and tilting modules corresponding to two-part partitions. Since parts of the proof are done by using induction, we first list the Weyl modules (and their structure) with highest weight $r < p^2$. We then list simple and uniserial Weyl modules. In a third step we prove that no other Weyl modules are uniserial. A tilting module is filtered by Weyl modules. Hence it can only be uniserial if all Weyl modules in its filtration are uniserial. In the fourth step we classify the uniserial tilting modules.

The submodule structure of the Weyl modules for type A_1 is described in [2] (without proof) and it can also be obtained from [7], so one might alternatively deduce the classification of uniserial Weyl modules from these references.

A uniserial module U has a unique composition series. We therefore introduce the following notation: If U has the composition series $U=U_0 > U_1 > \cdots > U_{n-1} > U_n = 0$ such that $U_{i-1}/U_i \cong L_i$, where L_i is simple and where $1 \leqslant i \leqslant n$, then we write $U=[L_1,\ldots,L_n]$. Recall that $\mathrm{Ext}^1_{S(2,r)}(L(t),L(s)) \cong \mathrm{Ext}^1_{S(2,r)}(L(s),L(t))$; and for $s<t$, this is non-zero if and only if $L(s)$ is a composition factor in the head of $\mathrm{rad}(\Delta(t))$. We denote the head or top of a module M by $\mathrm{hd}(M)$. We will use frequently the following fact, or its dual: A module U is uniserial if and only if it has a simple socle and the socle quotient is uniserial.

Example 2.1. It is well known that $\Delta(a)=L(a)$ for $\leqslant a \leqslant p-1$. If $r=ap+p-1$ with $1 \leqslant a \leqslant p-1$ then $\Delta(r)=\Delta(a)^F \otimes L(p-1)=L(r)$. If $r=ap+j$ with $0 \leqslant j \leqslant p-2$

then $\Delta(r) = [L(r), L(r-2-2j)]$, using Section 1.4 and the fact that Weyl modules corresponding to two-part partitions have a simple socle.

2.2. We now list the simple and uniserial Weyl modules. The following lemma is well known and follows by applying Section 1.4, by induction and Steinberg's tensor product theorem.

Lemma 2.1. *The module $\Delta(r)$ is simple if and only if $r = ap^t - 1$, where a,t are non-negative integers with $1 \leqslant a \leqslant p-1$.*

We consider in the following Weyl modules corresponding to $r = ap^t - 1 + b$ and $r = ap^t - 1 - b$ where a,b,t are non-negative integers with $1 \leqslant a,\ b \leqslant p-1$. Because of the examples listed in Example 2.1 we can assume that $t \geqslant 2$.

Proposition 2.1. *Let a,b,t be non-negative integers with $1 \leqslant a,\ b \leqslant p-1$. Then the module $\Delta(r)$ with $r = ap^t - 1 + b$ is uniserial. For $t \geqslant 2$ it has length $t+1$ and its structure is given by*

$$\Delta(r) = [L(r), L(r-2p^{t-1}), \ldots, L(r-2p^i), \ldots, L(r-2p), L(r-2b)],$$

where $t-1 \geqslant i \geqslant 1$. Let $r = ap^t - 1 - b$ and let $a-1-b \geqslant 0$ if $t = 0$. Then the module $\Delta(r)$ is uniserial. For $t \geqslant 2$ it has length t or $t+1$ and if $a = 1$ the structure is given by

$$\Delta(r) = [L(r), L(r-2p+2b), \ldots, L(r-2p^i+2b), \ldots, L(r-2p^{t-1}+2b)],$$

where $1 \leqslant i \leqslant t-1$; if $a > 1$ then the structure of $\Delta(r)$ is given by the same composition series, but extended by $L(r-2p^t+2b)$, as the socle.

Proof. If $t = 0$ or 1 for the values r given above, then the modules $\Delta(r)$ are uniserial (with the structure given in the above examples).

We will give the proof in the second case; the first one is similar. We consider $r = ap^t - 1 - b$ for $t \geqslant 2$; by 1.4 we have that $\Delta(r)$ is filtered by $\Delta(ap^{t-1}-1)^F \otimes L(p-b-1)$ and $\Delta(ap^{t-1}-2)^F \otimes L(b-1)$. By Lemma 2.1, the Weyl module $\Delta(ap^{t-1}-1)$ is simple and hence $\Delta(ap^{t-1}-1)^F \otimes L(p-b-1) \cong L(r)$ is the head of $\Delta(r)$. By the inductive hypothesis, the module $\Delta(ap^{t-1}-2)$ is uniserial and hence so is $\Delta(ap^{t-1}-2) \otimes L(b-1)$ (see Section 1.3). It follows that $\Delta(r)$ is uniserial.

The module $\Delta(ap^t-2)$ is an extension of $\Delta(ap^{t-1}-2)^F$ with $L(ap^t-2)$. We use induction on t. The Weyl module $\Delta(ap^t-2)$ is given by

$$\Delta(ap^t-2) = \begin{cases} [L(ap^t-2), \ldots, L(ap^t-2p^i), \ldots, L(ap^t-2p^{t-1})] & \text{if } a = 1, \\ [L(ap^t-2), \ldots, L(ap^t-2p^i), \ldots, L(ap^t-2p^t)] & \text{if } a > 1, \end{cases}$$

where $0 \leqslant i \leqslant t-1$ if $a = 1$ and where $0 \leqslant i \leqslant t$ otherwise. More general, the module $\Delta(r)$ for $r = ap^t - 1 - b$ is given by

$$\Delta(r) = \begin{cases} [L(r), L(r-2p+2b), \ldots, L(r-2p^i+2b), \ldots, L(r-2p^{t-1}+2b)], \\ [L(r), L(r-2p+2b), \ldots, L(r-2p^i+2b), \ldots, L(r-2p^t+2b)] \end{cases}$$

in the first case with $1 \leqslant i \leqslant t-1$ and $a=1$, and in the second case with $1 \leqslant i \leqslant t$ and $a > 1$. $\square$

We obtain more uniserial Weyl modules by twisting the above ones and tensoring with the Steinberg module (see Section 1.3). We list those in the following corollary.

Corollary 2.1. *Let a, b, t, k be non-negative integers with $1 \leqslant a,\ b \leqslant p-1$, let $r' = ap^t - 1 + b$ or let $r' = ap^t - 1 - b$ with $a-1-b \geqslant 0$ if $t=0$. Then $\Delta(r)$ with $r = r'p^k + p^k - 1$ is uniserial. Furthermore, if the structure of $\Delta(r')$ is given by $\Delta(r') = [L(t_1), \ldots, L(t_s)]$ then $\Delta(r) = [L(t_1 p^k + p^k - 1), \ldots, L(t_s p^k + p^k - 1)]$.*

2.3. In the following, we prove that no other Weyl modules than the ones listed so far are uniserial. To do so, we will reduce the problem to certain natural numbers r, for which we show in the next lemma that their corresponding Weyl module cannot be uniserial.

Lemma 2.2. *Let a, b, i, t be integers with $1 \leqslant a \leqslant p-1$, with $2 \leqslant b \leqslant p-1$, with $0 \leqslant i \leqslant p-2$ and $t \geqslant 2$. Then the Weyl module $\Delta(r)$ with $r = ap^{t+1} + (b-1)p + i$ is not uniserial. Similarly, let a, i, t be as above and let $1 \leqslant b \leqslant p-2$. Then $\Delta(r)$ with $r = ap^{t+1} - (b+1)p + i$ is not uniserial.*

Proof. We prove the second statement (the proof of the first part is easier and is similar). Let $r = ap^{t+1} - (b+1)p + i$ and assume that $\Delta(r)$ is uniserial. By Section 1.4, $\Delta(r)$ is filtered by $M_1 := L(i) \otimes \Delta(ap^t - 1 - b)^F$ and $M_2 := L(p-2-i) \otimes \Delta(ap^t - b - 2)^F$, where M_2 is a submodule. Since $\Delta(r)$ is uniserial, both modules in the filtration are uniserial such that

$$\operatorname{soc}(M_1) = \begin{cases} L((p-2)p^t + (b-1)p + i) & \text{if } a = 1, \\ L((a-2)p^{t+1} + (b-1)p + i) & \text{if } a > 1, \end{cases}$$

extends $\operatorname{hd}(M_2) = L(ap^{t+1} - (b+1)p - (i+2))$. Hence $\operatorname{soc}(M_1)$ occurs in the head of $\operatorname{rad}(\Delta(ap^{t+1} - (b+1)p - (i+2)))$. By Section 1.4, the module $N := \Delta(ap^{t+1} - (b+1)p - (i+2))$ is filtered by $N_1 := L(p-2-i) \otimes \Delta(ap^t - b - 2)^F$ and $N_2 := L(i) \otimes \Delta(ap^t - b - 3)^F$, where N_2 is isomorphic to a submodule. By Proposition 2.1 and Corollary 2.1, both N_1 and N_2 are uniserial (N_2 is uniserial for $b = p-2$ by Corollary 2.1) and hence the head of $\operatorname{rad}(N)$ contains at most $\operatorname{hd}(N_2) = L(ap^{t+1} - (b+3)p + i)$ and

$$\operatorname{hd}(\operatorname{rad}(N_1)) = L(ap^{t+1} + (b-1)p - 2p^2 - 2 - i).$$

Hence $\operatorname{soc}(M_1)$ does not occur in the head of $\operatorname{rad}(N)$, which contradicts the assumption. $\square$

Proposition 2.2. *The Weyl module $\Delta(r)$ is uniserial if and only if $r = ap^t - 1$ or $r = (ap^t - 1 + b)p^k + p^k - 1$ or $r = (ap^t - 1 - b)p^k + p^k - 1$ where $1 \leqslant a,\ b \leqslant p-1$, and where a, b, t, k are non-negative integers.*

Proof. One implication of the claim has been shown in Proposition 2.1 combined with Corollary 2.1. We now prove the other implication. With the examples given at the beginning of this section we can assume, without loss of generality, that $t \geqslant 2$. Let r be a natural number which is not of the above form and assume $\Delta(r)$ is uniserial. Using induction on r we find a contradiction. As induction hypothesis we assume that for all $\tilde{r} < r$ the claim has been established. Then there exists non-negative integers i, n such that $r = np + i$ where $0 \leqslant i \leqslant p-1$.

If $i \leqslant p-2$ then, by Section 1.4, the module $\Delta(r)$ is filtered by $L(i) \otimes \Delta(n)^F$ and $L(p-2-i) \otimes \Delta(n-1)^F$. Hence $\Delta(n)$ and $\Delta(n-1)$ both are uniserial, which by the inductive hypothesis happens if and only if n and $n-1$ both are equal to either $ap^t - 1$, $(ap^t - 1 - b)p^k + p^k - 1$ or $(ap^t - 1 + b)p^k + p^k - 1$ with $1 \leqslant a$, $b \leqslant p-1$ and $t, k \geq 0$. Since both n and $n-1$ have this special form, we obtain $k = 0$. Hence $r = np + i$ is equal to either $ap^{t+1} - (p-i)$, $ap^{t+1} - (b+1)p + i$ or $ap^{t+1} + (b-1)p + i$. In the first case this contradicts the assumption about the form of r, in the other two cases we either get a contradiction to the form of r (for $b = 1$ and $p-1$, respectively) or, applying Lemma 2.2, we get a contradiction to the assumption that $\Delta(r)$ is uniserial. If $i = p-1$ then one easily gets a similar contradiction. □

2.4. We are now ready to determine uniserial tilting modules. Let r be a natural number. Note that the tilting module $T(r)$ is simple if and only if $r = ap^t - 1$, where a, t are non-negative integers with $1 \leqslant a \leqslant p-1$. There exist unique natural numbers k and r' such that $r = p^k - 1 + p^k r'$ where $r' \not\equiv -1 \bmod p$, and $r' \geqslant 0$.

Proposition 2.3. *The tilting module $T(r)$ is uniserial if and only if either $r' \leqslant p-2$, or $r' = p^t + b - 1$ where $t \geqslant 1$ and $1 \leqslant b \leqslant p-1$. If $T(r)$ is uniserial and not simple, then its Δ-quotients are $\Delta(r)$ and $\Delta(p^k(p^t - 1 - b) + p^k - 1)$.*

Proof. Write $r = p^k - 1 + p^k r'$ where $r' \not\equiv -1 \bmod p$, then $T(r)$ is uniserial if and only if $T(r')$ is uniserial (see Section 1.3). They are simple iff $r' \leqslant p-2$.

If so then all Δ-quotients occurring in the Δ-filtration of $T(r')$ are uniserial, in particular so is $\Delta(r')$. Consider first the case when $r' = ap + j$ where $1 \leqslant a \leqslant p-1$ and $0 \leqslant j \leqslant p-2$. Then $T(r')$ has Δ-quotients with highest weights $(a-1)p + i$, $ap + j$ where $i + j = p-2$ (see Section 1.5). If $a \geqslant 2$ then this has length four and simple socle and head, and the middle is a direct sum. For $a = 1$, it has length three and is uniserial.

Now consider $r' \geqslant p^2$. Then by Proposition 2.2, $r' = ap^t - 1 + b$ or $r' = ap^t - 1 - b$ where $1 \leqslant a$, $b \leqslant p-1$. Consider the first possibility for r'. Let $j = b-1$ and $i + j = p-2$. By Section 1.5, we have $T(r') = T(p+j) \otimes T(ap^{t-1} - 1)^F$ and $T(r')$ contains $\Delta(r')$ with quotient $\Delta(ap^t - 2 - j)$. Both these Weyl modules are uniserial; hence $T(r')$ is uniserial if and only if there is a non-split subquotient of the socle of $\Delta(ap^t - 2 - j)$ with $L(r')$. But $T(r')$ is self-dual, hence this is true if and only if the head of $\operatorname{rad} \Delta(r')$ is isomorphic to the socle of $\Delta(ap^t - 2 - j)$. By Proposition 2.1 this holds if and only if $a = 1$.

Let $r' = ap^t - 2 - j$. By Section 1.5, we have $T(r') = T(p+i) \otimes T(ap^{t-1} - 2)^F$, and this has a Δ-quotient $\Delta(ap^t - 2p + j)$. By Proposition 2.2, the Weyl module $\Delta(ap^t - 2p + j)$ is not uniserial and therefore $T(r')$ is not uniserial either. □

3. The quiver of a good subalgebra of $S(2, r)$

3.1. The quiver of an algebra has vertices labelled by the simple modules, and the number of arrows from a simple module L to a simple module L' is equal to the dimension of $\mathrm{Ext}^1_{S(2,r)}(L, L')$. Extensions for simple $SL(2,K)$-modules have been determined by various authors (see for example [3]) and they describe the quivers for the Schur algebras $S(2,r)$. For convenience, we will start in this section with providing a proof for the quiver of $S(2,r)$. This will be relevant to understand the quiver for a good subalgebra $eS(2,r)e$.

Theorem 3.1. *Let $t < s$. Then $\mathrm{Ext}^1_{S(2,r)}(L(s), L(t)) = K$ if $s = v + p^n(pm+i)$ and $t = v + p^n(p(m-1)+j)$ where $v < p^n$, $m \not\equiv 0 \bmod p$ and $0 \leqslant i, j$ with $i + j = p-2$, for some $n \geqslant 0$; otherwise $\mathrm{Ext}_{S(2,r)}(L(s), L(t)) = 0$.*

Alternatively, suppose $s = \sum_{k \geqslant 0} s_k p^k$ is the p-adic expansion of s and let $s > t$. Then $\mathrm{Ext}^1_{S(2,r)}(L(s), L(t)) = K$ if and only if $s - t = (2s_n + 2)p^n$ where $s_n \leqslant p-2$, provided $s_{n+1} \neq 0$. Otherwise $\mathrm{Ext}^1_{S(2,r)}(L(s), L(t)) = 0$. This follows directly from the recursive description:

Proposition 3.1. *Let $s = s_0 + ps'$ and $t = t_0 + pt'$ where $0 \leqslant s_0,\ t_0 \leqslant p-1$ and $s', t' \geqslant 0$. For $p > 2$ and $s > t$ we have*

$$\mathrm{Ext}^1_{S(2,r)}(L(s), L(t)) = \begin{cases} \mathrm{Ext}^1_{S(2,r)}(L(s'), L(t')) & \text{if } s_0 = t_0, \\ K & \text{if } s_0 + t_0 = p-2, \\ & \quad t' + 1 = s' \not\equiv 0 \bmod p, \\ 0 & \text{otherwise.} \end{cases}$$

If $p = 2$ and $s > t$ we have

$$\mathrm{Ext}^1_{S(2,r)}(L(s), L(t))$$
$$= \begin{cases} \mathrm{Ext}^1_{S(2,r)}(L(s'), L(t')) & \text{if } s_0 = t_0 = 1 \text{ or} \\ & \text{if } s_0 = t_0 = 0 \text{ and } s' \equiv t' \bmod 2, \\ K & \text{if } s_0 = t_0 = 0,\ t' + 1 = s' \not\equiv 0 \bmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We occasionally need to distinguish in the following between $p > 2$ and $p = 2$. Therefore, we assume throughout the proof that $p > 2$. Modifications in case $p = 2$ are given in parenthesis.

(a) Recall that for $s > t$ the dimension of $\mathrm{Ext}^1_{S(2,r)}(L(s), L(t))$ is equal to the multiplicity of $L(t)$ as a composition factor in the head of the radical of $\Delta(s)$ (and we know this is here $\leqslant 1$). From Section 1.4 we see that this is zero unless $s_0 = t_0$ or $s_0 + t_0 = p - 2$. In case $s_0 = p - 1$, then $\Delta(s) \cong \Delta(s')^F \otimes L(p-1)$ and the stated result follows from Section 1.3. Now assume $0 \leqslant s_0 \leqslant p-2$, set $s_0 = i$, and $j = p - 2 - i$. By Section 1.4 it follows that the head of $\operatorname{rad} \Delta(s)$ is contained in $\operatorname{hdrad}(\Delta(s')^F \otimes L(i)) \oplus L(s'-1)^F \otimes L(j)$. If $t_0 = i$ then we get the stated reduction. Suppose now $t_0 = j$ and write $m = s'$. We will show in (b) that

$$\operatorname{Ext}^1_G(L(m)^F \otimes L(i), L(m-1)^F \otimes L(j)) \cong \begin{cases} 0 & \text{if } m \equiv 0 \bmod p, \\ K & \text{otherwise.} \end{cases} \tag{1}$$

This completes the proof of the proposition. $\square$

(In case $p = 2$, the only difference in the proof occurs when $s_0 = t_0 = 0$ and when we consider the exact sequence in Section 1.4. If $t' \equiv s' \bmod 2$ then we get the stated reduction, otherwise set $m = s'$ and continue as in the proof for $p > 2$.)

(b) Let $V = (L(m)^F \otimes L(i))^* \otimes L(m-1)^F \otimes L(j)$. Then by the 5-term sequence from the Lyndon–Hochschild–Serre spectral sequence we get

$$\begin{aligned} &0 \to H^1(G, (V^{G_1})^{-1}) \to H^1(G, V) \to H^1(G_1, V)^G \\ &\quad \to H^2(G, (V^{G_1})^{-1}) \to H^2(G, V). \end{aligned}$$

We will first show that the first and the last term of this sequence is zero. In a second step we then evaluate $H^1(G, V)$.

(i) We have V^{G_1} is isomorphic to $\operatorname{Hom}_{G_1}(L(i), L(j)) \otimes (L(m)^*)^F \otimes L(m-1)^F$. Assume first that $p > 2$, then $L(i)$, $L(j)$ are non-isomorphic and simple as G_1-modules and V^{G_1} is zero. It follows that the first and last term of the above sequence are zero.

(If $p = 2$, then $i = j = 0$ and the module is isomorphic to $L(m)^{*F} \otimes L(m-1)^F$, and we deduce $(V^{G_1})^{(-1)} \cong L(m)^* \otimes L(m-1)$. It follows that the first and last term of the sequence are isomorphic to $\operatorname{Ext}^i_G(L(m), L(m-1))$ for $i = 1, 2$ which is zero since the modules lie in different blocks.)

So $H^1(G, V) \cong H^1(G_1, V)^G$ which is isomorphic to

$$[\operatorname{Ext}^1_{G_1}(L(i), L(j)) \otimes (L(m)^* \otimes L(m-1))^F]^G.$$

(ii) We have the exact sequence $0 \to \Delta(p+j) \to T(p+j) \to L(i) \to 0$, which is a projective cover of $L(i)$ as a module for G_1. Applying $\operatorname{Hom}_{G_1}(-, L(j))$ gives

$$0 \to \operatorname{Hom}_{G_1}(\Delta(p+j), L(j)) \to \operatorname{Ext}^1_{G_1}(L(i), L(j)) \to 0.$$

(Note that this also holds for $p = 2$, in that case the first two terms of the long exact sequence are isomorphic.) As a G_1-module, $\Delta(p+j)$ has head isomorphic to $L(1)^F \otimes L(j)$ which shows that $\operatorname{Hom}_{G_1}(\Delta(p+j), L(j)) \cong \operatorname{Hom}_{G_1}(L(1)^F \otimes L(j), L(j))$, which is isomorphic to $(L(1)^*)^F$ as a G-module. So we get now

$$H^1(G_1, V)^G \cong \operatorname{Hom}_G((L(1) \otimes L(m))^F, L(m-1)^F).$$

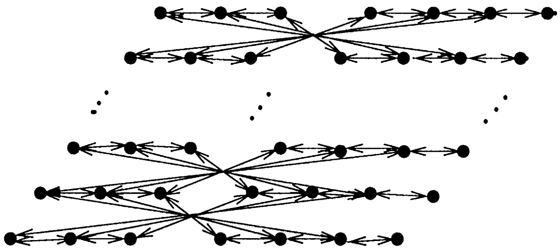

Fig. 1. The star-like pattern in the quiver of $S(2,r)$.

Assume first $m = pw$ say then $L(m) \cong L(w)^F$ and $L(1) \otimes L(m) \cong L(m+1)$ and the homomorphism space is zero. Otherwise $m = m_0 + pm'$ and $1 \leqslant m_0 \leqslant p-1$. We have

$$L(1) \otimes L(m_0) \cong \begin{cases} L(m_0+1) \oplus L(m_0-1), & m_0 \leqslant p-2, \\ T(p), & m_0 = p-1. \end{cases}$$

In the first case we get $L(1) \otimes L(m)$ is the direct sum of $L(m+1)$ and $L(m-1)$, and in the second case $L(1) \otimes L(m)$ has simple head $L(p-2) \otimes L(m')^F = L(m-1)$. So in both cases we get that the homomorphism space is one-dimensional, as stated, and Eq. (1) follows. $\square$

Remark. The expression for the extensions in the Schur algebra in Theorem 3.1 gives immediately that the quiver can be represented as a three-dimensional geometrical figure. Using [10], Theorem 13, it is enough to understand the quiver of the principal block. Such a quiver is illustrated in Fig. 2 for $S(2,100)$ in prime characteristic 3. The vertices in the graph are labelled increasingly by the weights in the block. Each horizontal layer consists of a square with p^2 vertices, and the edges in a horizontal layer are given by the star-like pattern as in Fig. 1. As vertical edges we only have two different types as indicated in Fig. 2.

3.2. We have seen that the extension of $L(s)$ (for $s = pm + i$) and $L(t)$ behave different, depending whether p divides m or not. We now study Weyl modules $\Delta(s)$ with $s = wp^b + i$ for $b \geqslant 1$ in order to understand possible new extensions of simples over $eS(2,r)e$.

Lemma 3.1. *Let $s = wp^b + i$ where p does not divide w, and $0 \leqslant i \leqslant p-2$, $i + j = p-2$ and $b \geqslant 1$. Then the Weyl module $\Delta(wp^b + i)$ has a filtration with quotients*

$$\begin{array}{rcl} \Delta(w)^{F^b} & \otimes & L(i) \\ \Delta(w-1)^{F^b} & \otimes & L(p^b - 2p^{b-1} + i) \\ \Delta(w-1)^{F^b} & \otimes & L(p^b - 2p^{b-2} + i) \\ & \cdots & \\ \Delta(w-1)^{F^b} & \otimes & L(p^b - 2p + i) \\ \Delta(w-1)^{F^b} & \otimes & L(p^b - p + j). \end{array}$$

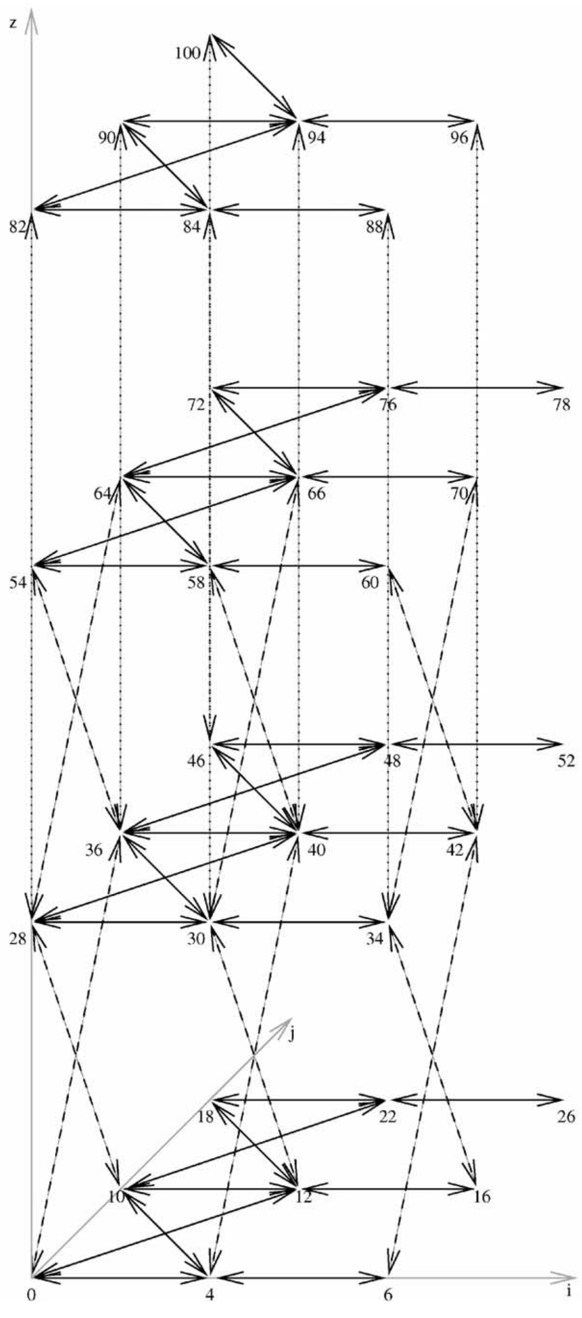

Fig. 2. The Quiver of the principal block of $S(2,100)$ for $p=3$.

Proof. By Section 1.4 we have the exact sequence

$$0 \to \Delta(wp^{b-1}-1)^F \otimes L(j) \to \Delta(s) \to \Delta(wp^{b-1})^F \otimes L(i) \to 0.$$

If $b=1$ then the claim follows. Assume now $b>1$ and proceed by induction on b (with arbitrary i). Note that $\Delta(wp^{b-1}-1) \cong \Delta(w-1)^{F^{b-1}} \otimes L(p^{b-1}-1)$, and hence the kernel of this sequence is isomorphic to $\Delta(w-1)^{F^b} \otimes L(p^{b-1}-p+j)$. The inductive hypothesis gives a filtration for the cokernel in the exact sequence, and the statement follows directly. □

Definition. Let $s=up+i \geqslant p$ where $0 \leqslant i \leqslant p-2$, $i+j=p-2$ and $u \geqslant 1$. Moreover, let U be a quotient module of $\Delta(s)$. We say that U is a *special quotient* if it has simple socle isomorphic to $L((u-1)p+j)$. This means that the socle of U is the top composition factor of the submodule in the short exact sequence in Section 1.4.

Suppose $s=up+i$ with $u \not\equiv 0 \bmod p$, then $L((u-1)p+j)$ occurs in the head of $\operatorname{rad}\Delta(s)$ by Proposition 3.1. Hence $\Delta(s)$ has a special quotient of length two, head $L(s)$ and socle $L((u-1)p+j)$ and no other special quotient. Next, consider $s=up+i$ with $up=wp^t$ where $t \geqslant 1$ and w is not divisible by p. The following classifies special quotients in the given situation.

Lemma 3.2. *Let $s=up+i=wp^b+i$ with $i \leqslant p-2$ and where p and w are coprime, $w \neq 0$ and let $b \geqslant 1$. Then $\Delta(s)$ has a unique special quotient, and this is uniserial of length $b+1$ and has the form $[L(s), L(s-2p^{b-1}), \ldots, L(s-2p), L(s-(2i+2))]$.*

Proof. The proof is by induction on b. Consider the top composition factors of the quotients in the filtration in Lemma 3.1, say $L=L(w-1)^{F^b} \otimes L(p^b-p+j)$, or then successively $L=L(w-1)^{F^b} \otimes L(p^b-2p^{b-c}+i)$, where $1 \leqslant c \leqslant b-1$. Since Weyl modules have a simple top there must be a simple module $\tilde{L}$ which is a composition factor in some higher quotient, such that $\Delta(s)$ has an indecomposable subquotient with top $\tilde{L}$ and socle L. By Theorem 3.1, the only composition factor which has non-split extensions with this simple L is the top composition factor of the next quotient.

Now use the following general argument, to prove the existence: Let L be a simple module and let $0 \to L \to M \to V \to 0$ be an exact sequence such that V has a quotient U' with simple socle L' and such that $\operatorname{Ext}^1_{S(2,r)}(L,\tilde{L})=0$ for all composition factors $\tilde{L}$ of V with $L' \neq \tilde{L}$ (and V is multiplicity-free). Then M has a quotient U with socle containing L and $U/L=U'$. Moreover if M has a simple head then $\operatorname{soc}(U)=L$. This is easy to prove and applying this to the above situation establishes the existence of a special quotient.

We will now prove uniqueness. For this we use that $\Delta(r)$ is multiplicity-free. If $b=1$ then $L(s-2-2i)$ occurs in the head of $\operatorname{rad}\Delta(s)$, so there is a special quotient of length two, it is unique, and there are no others. Now suppose that $b \geqslant 2$. Let U be a special quotient of $\Delta(s)$ with socle $L(t)$. By the argument as before, if U has length >2 (that is p divides u), then there is a unique composition factor $L(m)$ of

$\Delta(u)^F \otimes L(i)$ which has non-split extensions with $L(t)$. Hence $U/L(t)$ must have simple socle $L(m)$. Now use induction. $\square$

Let U be the special quotient of $\Delta(s)$ in Lemma 3.2. We observe that the weights of the composition factors of $\operatorname{rad} U$ are all $\leqslant s-(2i+2)$. Hence, by the universal property of Weyl modules, the uniserial module $(\operatorname{rad} U)^\circ$ (here $^\circ$ denotes the dual) must be a quotient of $\Delta(\tilde{s})$ where $\tilde{s}=s-(2i+2)$. We can write $\tilde{s}=\tilde{w}p^b+p^b-p+j$, where $0\leqslant j\leqslant p-2$ with $i+j=p-2$ and $\tilde{w}=w-1$. Then $\Delta(\tilde{s})$ has a filtration (use Section 1.4) with quotients

$$
\begin{array}{rcl}
\Delta(\tilde{w})^{F^b} & \otimes & L(p^b-p+j)\\
\Delta(\tilde{w})^{F^b} & \otimes & L(p^b-2p+i)\\
\Delta(\tilde{w})^{F^b} & \otimes & L(p^b-2p^2+i)\\
 & \cdots & \\
\Delta(\tilde{w})^{F^b} & \otimes & L(p^b-2p^{b-1}+i)\\
\Delta(\tilde{w}-1)^{F^b} & \otimes & L(i),
\end{array}
$$

where the last quotient only occurs if $\tilde{w}\neq 0$. The uniserial quotient $(\operatorname{rad} U)^\circ$ has composition factors the top composition factors of these quotients, except the last in case $\tilde{w}=0$ and $b=1$.

3.3. We study now the quiver of eSe where $S=S(2,r)$ with arbitrary r and where e is a good idempotent. So let e be of the form $e=e_\Gamma=\sum_{j\in\Gamma} e_j$ where $\Gamma=\{j\geqslant\tau \mid j\equiv r \bmod 2\}$, for some τ.

Then eSe has simple modules $eL(s)$ for $s\in\Gamma$, and is quasi-hereditary with standard modules $e\Delta(s)$. These modules are multiplicity-free. The algebra eSe also has a duality fixing the simple modules, and hence, as for S, we can determine the quiver from the Weyl modules: For $s>t\geqslant\tau$ the dimension of $\operatorname{Ext}^1_{eSe}(eL(s),eL(t))$ is one if $eL(t)$ occurs in the head of the radical of $e\Delta(s)$ and is zero otherwise.

The full subquiver of S whose vertices have labels $\geqslant\tau$ is contained in the quiver of eSe and is completely described by Section 3.1. It remains to describe any additional arrows for eSe. That is we need to find all $s,t\geqslant\tau$ where $\operatorname{Ext}^1_S(L(s),L(t))=0$ but $\operatorname{Ext}^1_{eSe}(eL(s),eL(t))\neq 0$.

Proposition 3.2. *Let $s>t\geqslant\tau$. Then there is a new arrow $s\to t$ in the quiver of eSe if and only if the following holds: $s=(pm+i)p^n+v$ and $t=(p(m-1)+j)p^n+v$ for $v<p^n$, and $0\leqslant i,j$ with $i+j=p-2$, moreover $m\equiv 0 \bmod p$ and $s-2p^{n+1}<\tau$, for some $n\geqslant 0$.*

Combining this with Theorem 3.1 we therefore have

Theorem 3.2. *Let $s\geqslant t\geqslant\tau$ and let $s=\sum_{k\geqslant 0} s_k p^k$ be the p-adic decomposition of s. Then $\operatorname{Ext}^1_{eSe}(eL(s),eL(t))$ is at most one dimensional, and it is non-zero if and only if*

$s-t=(2s_n+2)p^n$ where $s_n \leqslant p-2$, and either $s_{n+1} \neq 0$ or $s_{n+1}=0$ but $s-2p^{n+1} < \tau$, for some $n \geqslant 0$.

Proof. We start with a proof of Proposition 3.2. There is a new arrow $s \to t$ where $s > t$ if and only if $\Delta(s)$ has a quotient W which has a simple socle $L(t)$ and a simple head $L(s)$, and if $Y = \mathrm{rad}(W)/\mathrm{soc}(W)$ then $Y \neq 0$ and $eY = 0$.

(1) We assume we have a new arrow $s \to t$, that is there is such a quotient W. We claim that we can write $s=(pm+i)p^n+v$ and $t=(pz+j)p^n+v$ where $v < p^n$, and $pm+i > pz+j$, $i+j=p-2$ (hence $s_n = i \leqslant p-2$), and moreover that $W = M^{F^n} \otimes L(v)$, and M is a quotient of $\Delta(pm+i)$ but not of $\Delta(m)^F \otimes L(i)$. To see this one applies repeatedly the following two reductions:

(a) If $s=s'p+p-1$ for $s' \geqslant 0$ then $\Delta(s) \cong \Delta(s')^F \otimes L(p-1)$ and by 1.3 $W = W_1^F \otimes L(p-1)$, and moreover W_1 is a quotient of $\Delta(s')$. Since $L(t)$ is a composition factor of W we deduce that $t = t'p+p-1$. We continue with W_1.

(b) Suppose $s=s'p+i$ with $0 \leqslant i \leqslant p-2$, and $t=t'p+i$ with $t' \geqslant 0$. Consider the filtration of $\Delta(s)$ as in 1.4 with submodule $\Delta(s'-1)^F \otimes L(j)$, and quotient $\Delta(s')^F \otimes L(i)$. Assume first that $p \neq 2$, then $i \neq j$ and we must have that W is a quotient of $\Delta(s')^F \otimes L(i)$. Moreover, by 1.3 we have $W = W_1^F \otimes L(i)$ and W_1 is a quotient of $\Delta(s')$. Assume now that $p=2$. If W is a quotient of $\Delta(s')^F$, then $W = W_1^F$ where W_1 by Section 1.3 is a quotient of $\Delta(s')$ and continue with W_1. Otherwise we have reached the statement.

(2) Having established (1), let $W = M^{F^n} \otimes L(v)$, and let M be a quotient of $\Delta(pm+i)$ with socle $L(w)$ where $w = pz+j$ for some $z \geqslant 0$ and $i+j=p-2$. Let $L(g)$ be a composition factors of $Z := \mathrm{rad}(M)/\mathrm{soc}(M)$. By hypothesis, eY is zero. Hence $p^n g + v < \tau \leqslant t$ and so

$$g < pz+j = w. \tag{2}$$

Since $0 \to \Delta(m-1)^F \otimes L(j) \to \Delta(pm+i) \to \Delta(m)^F \otimes L(i) \to 0$ is exact and M is not a quotient of $\Delta(m)^F \otimes L(i)$, we have an exact sequence $0 \to M' \to M \to M'' \to 0$ where M' is a non-zero quotient of $\Delta(m-1)^F \otimes L(j)$ and M'' is a quotient of $\Delta(m)^F \otimes L(i)$. We claim that $M' = L(w)$. We have

$$0 \neq \mathrm{soc}(M') \subseteq \mathrm{soc}(M) = L(w),$$

so $L(w) = \mathrm{soc}(M')$. Since M' is a non-zero quotient of $\Delta(m-1)^F \otimes L(j)$, it has a simple top. Assume for a contradiction that M' is not simple. Then its top composition factor has highest weight so that $w < g$ for some composition factor $L(g)$ occurring in $M'/\mathrm{soc}(M')$. This contradicts Eq. (2). Hence $M' = L(w)$ and $z = m-1$. By the Definition in Section 3.2, M is a special quotient of $\Delta(pm+i)$. We so far have seen that $s-t=(2s_n+2)p^n$ with $s_n \leqslant p-2$. Since we consider a *new* arrow in the quiver of *eSe*, Theorem 3.1 implies that $s_{n+1}=0$. By Lemma 3.2, the composition factor of Z of lowest weight is $L(pm+i-2p)$, where by hypothesis $((pm+i)-2p)p^n+v < \tau$. This is equivalent to $s-2p^{n+1} < \tau$, as required.

(3) To prove the converse implication, we must show that $\Delta(s)$ has a quotient W with (simple head $L(s)$ and) simple socle $L(t)$ and whose other composition factors are all annihilated by e. Let U be the special quotient of $\Delta(pm+i)$ as in Lemma 3.2, and take $W = U^{F^n} \otimes L(v)$. Using the hypothesis that $s - 2p^{n+1} < \tau$ it follows that W has the required properties. $\square$

Example. Consider (as in the Remark in Section 3.1) the principal block of the Schur algebra $eS(2,r)e$ for $r=100$ and $p=3$ where e is given by τ as above. For $\tau=36$ we get no new arrows, for $\tau=40$ we get new arrows $42 \leftrightarrow 46$ and $48 \leftrightarrow 54$, for $\tau=42$ between $48 \leftrightarrow 54$. Compare with Fig. 2.

3.4. Assume $p=2$ and let $r=2^{k+1}-2$ be even. Then we also need to know the quiver of $fS(2,r)f$ where f is an idempotent which annihilates $L(r)$ but no other simple module. Note that f is not a good idempotent and hence we cannot determine the extensions of the simple modules of $fS(2,r)f$ from modules $f\Delta(m)$ as before.

The question is therefore whether there is a uniserial module U say of length three with middle $L(r)$, and where the head is $L(m)$ (say) with $m<r$ and the socle is $L(t)$ with $t<r$. Such modules correspond precisely to new arrows in the quiver of $fS(2,r)f$ which are not present in the quiver of $S(2,r)$.

Take such module U, then $\mathrm{rad}(U)$ is a quotient of $\Delta(r)$. For this particular degree $\Delta(r)$ is uniserial, and the head of $\mathrm{rad}\,\Delta(r)$ is isomorphic to $L(r-2)$ (see Proposition 2.1). So the head of $\mathrm{rad}\,f\Delta(r)$ is simple and isomorphic to $eL(r-2)$. Dually by viewing $U/\mathrm{soc}(U)$ as a submodule of $\nabla(r)$ we get that the only possibility for U is with socle and head $L(r-2)$, and hence there is precisely one new arrow in the quiver of $fS(2,r)f$ and this is a loop at vertex $r-2$.

4. Uniserial Weyl modules and tilting modules for $eS(2,r)e$

In analogy to Section 2 we classify now the uniserial Weyl modules and tilting modules for $eS(2,r)e$ where, as in the previous section, e is of the form $e=e_\Gamma=\sum_{j\in\Gamma} e_j$ with $\Gamma=\{j \geqslant \tau \mid j \equiv r \bmod 2\}$, for some τ. For a natural number $0 \leqslant s \leqslant p-2$ we define $\hat{s}=p-2-s$.

4.1. We begin with the classification of the uniserial Weyl modules of $eS(2,r)e$. Since the weights for $S(2,r)$ are linearly ordered and since Weyl modules are multiplicity-free, the radical of $\Delta(s)$ has a highest weight, which we describe in the following lemma.

Lemma 4.1. *Let $s=(s_m,\ldots,s_k,p-1,\ldots,p-1)$ where $s_k \neq p-1$ and $k \geqslant 0$. Then the highest weight of the radical of $\Delta(s)$ is $s-(2+2s_k)p^k$, unless $\Delta(s)$ is simple.*

Proof. Let $\tilde{s}=(s_m,\ldots,s_k)$. If $\tilde{s}<p$ then $\Delta(\tilde{s})$ and $\Delta(s)$ are simple. Otherwise, by Section 1.4 the highest weight of the radical of $\Delta(\tilde{s})$ is $\tilde{s}-(2+2s_k)$, and the claim follows since $\Delta(s) \cong \Delta(\tilde{s})^{F^k} \otimes L(p^k-1)$. $\square$

Proposition 4.1. *Let* $s=(s_m,\dots,s_k,p-1,\dots,p-1)$ *where* $s_k\neq p-1$ *and* $k\geqslant 0$. *Let* $t=s-(2+2s_k)p^k$. *Moreover, for parts* (b) *and* (c) *we assume* $e\Delta(s)$ *is non-zero and not simple, so* $\tau\leqslant t$.

(a) *The module* $e\Delta(s)$ *is simple if and only if* $t<\tau\leqslant s$.

(b) *Suppose* $s_{k+1}\neq p-1$ *and let* $b\geqslant 1$ *be minimal such that* $s_{k+b}\neq 0$. *Let* $\beta=s-2p^{k+1}-2s_{k+b}p^{k+b}$. *Then* $e\Delta(s)$ *is uniserial if and only if* $\beta<\tau$. *If so then it has length* $\leqslant b+2$, *and the structure is given by*

$$[eL(s),eL(s-2p^{b+k-1}),\dots,eL(s-2p^i),\dots,$$
$$eL(s-2p^{k+1}),eL(t),eL(t-2s_{k+b}p^{k+b})]$$

(*for* $k+1\leqslant i\leqslant b+k-1$) *where some of these may be zero.*

(c) *Suppose* $s_{k+1}=p-1$ *and let* $\geqslant 2$ *be minimal such that* $s_{k+c}\neq p-1$. *Let* $\gamma=s-(2+2s_{k+c})p^{k+c}$. *Then* $e\Delta(s)$ *is uniserial if and only if* $s-(2+2s_{k+c}p^{k+c})<\tau$. *If so then it has length* $\leqslant c+2$ *and the structure is given by*

$$[eL(s),eL(t),eL(t-(2p-2)p^{k+1}),\dots,eL(t-(2p^i-2)p^{k+1}),\dots,$$
$$eL(t-(2p^{c-1}-2)p^{k+1}),eL(t-(2p^{c-1}-2)p^{k+1}-2s_{k+c}p^{k+c})]$$

(*for* $1\leqslant i\leqslant c-1$) *where some of these may be zero.*

Proof. Part (a) follows from Lemma 4.1 (note that (a) is also true when $s=(p-1,\dots,p-1)$). Assume now that $e\Delta(s)$ is non-zero and not simple. Then, in particular, $\Delta(\tilde{s})$ is non-zero and not simple where $\tilde{s}=(s_m,\dots,s_k)$, and therefore $\tilde{s}\geqslant p$.

(b) Applying Lemma 3.1 (using $s_{k+1}\neq p-1$) to $\tilde{s}=wp^b+s_k$ we know that $\Delta(s)$ has a filtration with quotients

$$\begin{array}{ll}
\Delta(w)^{F^{b+k}} & \otimes\ L(p^ks_k+p^k-1)\\
\Delta(w-1)^{F^{b+k}} & \otimes\ L(p^{b+k}-2p^{k+b-1}+s_kp^k+p^k-1)\\
 & \dots\\
\Delta(w-1)^{F^{b+k}} & \otimes\ L(p^{b+k}-2p^{k+j}+s_kp^k+p^k-1)\\
 & \dots\\
\Delta(w-1)^{F^{b+k}} & \otimes\ L(p^{b+k}-2p^{k+1}+s_kp^k+p^k-1)\\
\Delta(w-1)^{F^{b+k}} & \otimes\ L(p^{b+k}-p^{k+1}+\hat{s}_kp^k+p^k-1),
\end{array} \tag{3}$$

where $w=(s_m,\dots,s_{k+b})$. Let Y be the lowest quotient, its top composition factor has highest weight $t=s-(2+2s_k)p^k$. Let U be the special quotient of $\Delta(s)$, and recall from Lemma 3.2 that it is uniserial, with socle $L(t)$. We assume that $e\Delta(s)$ is uniserial but not simple. Let $\beta=s-2p^{k+1}-2s_{k+b}p^{k+b}$.

(1) We claim that then $e(\Delta(s)/\mathrm{rad}(Y))=eU$. If $\pi:\Delta(s)\to U$ is an epimorphism then the kernel of π contains $\mathrm{rad}\,Y$, so we have an exact sequence

$$0\to X\to\Delta(s)/\mathrm{rad}(Y)\to U\to 0.$$

This gives an exact sequence $0 \to eX \to e(\Delta(s)/\mathrm{rad}(Y)) \to eU \to 0$ whose middle term is uniserial, since $e\Delta(s)$ is uniserial; so it has a simple socle. We have

$$\mathrm{soc}(eX) \subseteq \mathrm{soc}(e\Delta(s)/\mathrm{rad}(Y)) \subseteq \mathrm{soc}(eX \oplus eU).$$

But the socle of eU is $eL(t) \subseteq \mathrm{soc}(e\Delta(s)/\mathrm{rad}(Y))$ which is simple. So it follows that $\mathrm{soc}(eX) = 0$ and hence $eX = 0$.

(2) We will show that $\beta < \tau$. Suppose first that $\Delta(w-1)$ is simple. Then since $s_{k+b} \neq 0$, $w-1 \not\equiv -1 \pmod p$ and it follows that $w-1 \leqslant p-2$. So $w = s_{k+b}$ which implies $\beta < 0$ and hence $\beta < \tau$. So we assume from now that $\Delta(w-1)$ is not simple.

By (1) we have that e annihilates the radical of each of the quotients in the above filtration, except possibly for Y. If $L(g)$ is simple and $g < p^{b+k}$ then by Lemma 4.1 (using $s_{k+b} \neq 0$) we know that $\mathrm{rad}\,\Delta(w-1)^{F^{b+k}} \otimes L(g)$ has highest weight

$$((w-1)p^{b+k} + g) - 2s_{k+b}p^{k+b}. \tag{4}$$

The top weights of the quotients in filtration (3) (except for the highest) are in increasing order, from top to bottom. So e annihilates the radical of the quotient $\Delta(w-1)^{F^{b+k}} \otimes L(g)$ (other than Y) if and only if it annihilates the radical of the second lowest quotient. This has highest weight β and hence $\beta < \tau$.

(3) Assume $\beta < \tau$, we will show that then $e\Delta(s)$ is uniserial. We have already seen that then e annihilates the radicals of all quotients in filtration (3) except possibly the highest and lowest quotient. Consider the highest; if $\Delta(w)$ is not simple then the radical of the highest quotient in filtration (3) has highest weight $s-(2+2s_{b+k+l})p^{b+k+l}$ (see Lemma 4.1) where $l \geqslant 0$ is smallest such that $s_{k+b+l} \neq p-1$. This weight is $\leqslant \beta$, and hence e also annihilates the radical of the top quotient in filtration (3). Hence if $\beta < \tau$ then $e(\Delta(s)/\mathrm{rad}(Y)) = eU$.

(a) We will now show that eY has length $\leqslant 2$. (By Eq. (4), the highest weight of $\mathrm{rad}(Y)$ is $t - 2s_{k+b}p^{k+b} = \beta + 2p^{k+1}$ which may or may not be $< \tau$.) By Section 1.4, the module $\Delta(w-1)$ has a filtration with quotients

$$\Delta(w')^F \otimes L(i) \quad \text{and} \quad \Delta(w'-1)^F \otimes L(\hat{i}),$$

where $i = s_{k+b} - 1$ and $w' = (s_m, \ldots, s_{k+b+1})$. By Lemma 4.1, the radicals of these quotients (if non-zero) have highest weights g_1 and g_2 with

$$g_1 = (pw' + i) - (2 + 2s_{k+b+l})p^l,$$

$$g_2 = (p(w'-1) + \hat{i}) - 2s_{k+b+r}p^r,$$

where $l \geqslant 1$ is smallest such that $s_{k+b+l} \neq p-1$ and where $r \geqslant 1$ is smallest such that $s_{k+b+r} \neq 0$. Both the composition factors $L(g_1)$ and $L(g_2)$ of $\Delta(w-1)$ give rise to composition factors of Y, whose highest weights are

$$t - (2 + 2s_{k+b+l}p^{k+b+l}),$$

$$t - 2s_{k+b}p^{k+b} - 2s_{k+b+r}p^{k+b+r},$$

which are both smaller than β. Hence, eY has length at most two and composition factors $eL(t)$ and possibly $eL(t - 2s_{k+b}p^{k+b})$.

(b) If $eL(t-2s_{k+b}p^{k+b})$ is zero then $e\Delta(s)=eU$, which is uniserial. Suppose it is non-zero. By Theorem 3.2, it does not have non-split extensions with any composition factor of $e\Delta(s)$ other than $eL(t)$ and hence in this case $e\Delta(s)$ is uniserial as well.

(c) Now assume $s_{k+1}=p-1$. Let $w=(s_m,\ldots,s_{k+c})$, then by the Remark in Section 3.2 we know that $\Delta(s)$ has a filtration with quotients

$$\begin{aligned}
&\Delta(w)^{F^{c+k}}\otimes L(p^{k+c}-p^{k+1}+s_kp^k+p^k-1)\\
&\Delta(w)^{F^{c+k}}\otimes L(p^{k+c}-2p^{k+1}+\hat{s}_kp^k+p^k-1)\\
&\ldots\\
&\Delta(w)^{F^{k+c}}\otimes L(p^{k+c}-2p^{k+j}+\hat{s}_kp^k+p^k-1)\\
&\ldots\\
&\Delta(w)^{F^{k+c}}\otimes L(p^{k+c}-2p^{k+c-1}+\hat{s}_kp^k+p^k-1)\\
&\Delta(w-1)^{F^{k+c}}\otimes L(\hat{s}_kp^k+p^k-1).
\end{aligned}\tag{5}$$

Recall also that $\Delta(s)$ has a uniserial quotient V say whose composition factors are precisely the top factors of these quotients, except for the lowest one, and they are given as

$$L(s),\ L(t),\ L(t-(2p-2)p^{k+1}),\ldots,L(t-(2p^j-2)p^{k+1}),\ldots,$$
$$L(t-(2p^{c-2}-2)p^{k+1}).$$

Note that their highest weights are in decreasing order. We assume that $e\Delta(s)$ is uniserial and not simple. Let $\gamma=s-(2+2s_{k+c})p^{k+c}$.

(1) We claim that $\gamma<\tau$. By hypothesis $\tau\leqslant t$ and so we have $eL(t)\neq 0$. Moreover, $\mathrm{Ext}^1_{S(2,r)}(L(s),L(t))\neq 0$ and hence $e\Delta(s)$ has a uniserial quotient of length two with top $eL(s)$ and socle $eL(t)$. Similarly, as in (1) in the proof of (b), we see that e must annihilate the radical of the top quotient in (5). This has highest weight γ and hence $\gamma<\tau$.

(2) We claim that if $\gamma<\tau$ then $e\Delta(s)$ has composition factors as stated. If $\gamma<\tau$ then also $y-(2+2s_{k+c})p^{k+c}<\tau$ for $y=t$ and $y=t-(2p^j-2)p^{k+1}$ for $1\leqslant j\leqslant c-2$; these are the highest weights of the radicals of the quotients, other than the lowest one. So e annihilates these radicals. It remains to consider the lowest quotient, call it Z, and we must show that it has length $\leqslant 2$. This is clear if $\Delta(w-1)$ is simple. Otherwise the radical of Z has highest weight

$$t-(2p^{c-1}-2)p^{k+1}-2s_{k+c+r}p^{k+c+r},$$

where $r\geqslant 0$ is minimal such that $s_{k+c+r}\neq 0$. If $r\geqslant 1$ then this is $<\gamma$.

So assume now $r=0$, that is $s_{k+c}\neq 0$, in which case the weight need not be $<\gamma$. Consider $\Delta(w-1)$, it has a filtration with quotients $\Delta(w')^F\otimes L(i)$ and $\Delta(w'-1)^F\otimes L(\hat{i})$ where $w-1=pw'+i$ and $i=s_{k+c}-1$. The highest weights h_1 and h_2 of the radical of $\Delta(w')$ and $\Delta(w'-1)$, respectively, are

$$h_1=w'-(2+2s_{k+c+l})p^{l-1},$$
$$h_2=w'-1-2s_{k+c+r}p^{r-1},$$

where $l \geqslant 1$ is smallest such that $s_{k+c+l} \neq p-1$ and where $r \geqslant 1$ is smallest such that $s_{k+c+r} \neq 0$. The corresponding weights of a composition factor of Z are

$$t - 2p^{k+c} + 2p^{k+1} - (2 + 2s_{k+c+l})p^{k+c+l},$$

$$t - 2p^{k+c} + 2p^{k+1} - 2s_{k+c}p^{k+c} - 2s_{k+c+r}p^{k+c+r},$$

which are both $< \gamma$ and e annihilates the radical of both the quotient. So eZ has at most two composition factors.

(3) We know already that $eV = e\Delta(s)/eZ$ is uniserial. For all possibilities of eZ, the module $e\Delta(s)$ is uniserial, by the argument as in the proof of (b). □

4.2. We will now classify uniserial tilting modules for $eS(2,r)e$. Recall that these are the modules $eT(s)$ with $\tau \leqslant s$, and that $eT(s)$ has Δ-quotients precisely those $e\Delta(v)$ with $\Delta(v)$ occurs in $T(s)$ and where $\tau \leqslant v$. The following remark is used in the proof of Proposition 4.2(i), part (1).

Remark. We consider the order of the weights v such that $\Delta(v)$ occurs in $T(s)$; using Sections 1.4 and 1.5 we have

(a) If $s = pw + p - 1$ then $T(s) \cong T(w)^F \otimes L(p-1)$ and it has Δ-quotients of the form $\Delta(t)^F \otimes L(p-1)$, and $t \to pt + p - 1$ preserves the order.

(b) Suppose $s = pw + i$ where $0 \leqslant i \leqslant p-2$; then $T(s) \cong T(p+i) \otimes T(w-1)^F$. If $\Delta(t_1)$ and $\Delta(t_2)$ occur in $T(w-1)$ and $t_1 < t_2$ then (since $t_1 + 1 < t_2$, as t_1, t_2 have the same parity) the weights of the Δ-quotients of $T(p+i) \otimes \Delta(t_j)^F$ satisfy

$$pt_1 + \hat{i} < p(t_1+1) + i < pt_2 + \hat{i} < p(t_2+1) + i.$$

Proposition 4.2. *Let $s = (s_m, \ldots, s_k, p-1, \ldots, p-1)$ with $s_k \neq p-1$ where $k \geqslant 0$ and let $t = s - (2 + 2s_k)p^k$. The tilting module $eT(s)$ is uniserial and not simple if and only if $\tau \leqslant t$ and one of the following holds*:

(a) *We have $s_{k+1} \neq p-1$ and $g < \tau$ where $g = t - 2s_{k+b}p^{k+b}$ and $b \geqslant 1$ is minimal such that $s_{k+b} \neq 0$.*

(b) *We have $s_{k+1} = p-1$ and $s - 2p^{k+1} < \tau$. Here $eT(s)$ has length three.*

In both cases $eT(s)$ has two Δ-quotients, $\Delta(s)$ and $\Delta(t)$.

Proof. Assume $eT(s)$ is uniserial and not simple, then it follows that $e\Delta(s)$ also is uniserial and not simple. Hence s must satisfy (b) or (c) in Proposition 4.1. Since $eT(s)$ is self-dual we must have $e(T(s)/\mathrm{rad}\,\Delta(s)) \cong (e\Delta(s))^{\circ}$. Hence the composition series of $eT(s)$ is completely determined by that of $e\Delta(s)$.

(a) Assume first that $s_{k+1} \neq p-1$, so that $e\Delta(s)$ is given in Proposition 4.1(b); then for $\beta := s - 2p^{k+1} - 2s_{k+b}p^{k+b}$ we have $\beta < \tau \leqslant t$. Let $g := t - 2s_{k+b}p^{k+b}$. The order on the highest weights of the composition factors of $e\Delta(s)$ is

$$s > t > s - 2p^{k+1} > \cdots > s - 2p^{i} > \cdots > s - 2p^{b+k-1} > g$$

and we know that $eL(t) \neq 0$. So $eL(t)$ is the composition factor with highest weight in $eT(s)/e\Delta(s)$ and hence $e\Delta(t)$ must occur in $eT(s)$.

(1) We claim that $g < \tau$. We know (by Section 1.5) that $T(s) = T(\tilde{s})^{F^k} \otimes \Delta(p^k - 1)$ and $T(\tilde{s}) = T(p + s_k) \otimes T(w-1)^{F^b} \otimes T(p^{b-1} - 1)^F$, here $\tilde{s} = (s_m, \ldots, s_{k+b}, 0, \ldots, 0, s_k)$ with $s_{k+b} \neq 0$ and $w = (s_m, \ldots, s_{k+b})$.

If $T(w-1)$ is simple then we are done. So assume $T(w-1)$ is not simple, then it is isomorphic to $T(p+i) \otimes T(w'-1)^F$ where $w' = (s_m, \ldots, s_{b+k+1})$ and $i = s_{k+b} - 1$, and hence the two Δ-quotients of $T(w-1)$ (arising from $T(p+i) \otimes \Delta(w'-1)^F$) with highest weights are $\Delta(v)$ and $\Delta(w-1)$, where $v = p(w'-1) + \hat{\imath}$. The Δ-quotients arising from $\Delta(w-1)$ are $\Delta(s)$ and $\Delta(t)$. The module $\Delta(v)$ gives rise to Δ-quotients of $T(s)$ with highest weights g and $\tilde{g}$ with $g < \tilde{g} = s - 2s_{k+b}p^{k+b}$. Assume for a contradiction that $\tau \leqslant g$. Then also $\tau \leqslant \tilde{g}$. By the above $eT(s)$ has also $e\Delta(\tilde{g})$ as a quotient; hence $eL(\tilde{g})$ must be a composition factor of $e\Delta(s)$, which is (by Proposition 4.1) not the case. So $g < \tau$. Moreover, by the above remark on the order of weights, the only Δ-quotients occuring in $eT(s)$ are $e\Delta(s)$ and $e\Delta(t)$.

(2) We claim that $eT(s)$ has quotients $e\Delta(s)$ and $e\Delta(t)$ and is uniserial. We have already seen in (1) that the Δ-quotients $e\Delta(s)$ and $e\Delta(t)$ occur in $eT(s)$ and no others. We know that $e\Delta(s)$ is uniserial. Moreover, by Proposition 4.1(b) or (c), we obtain that $e\Delta(t)$ is uniserial. By arguments as in Section 2 it follows that $eT(s)$ is uniserial.

(b) Now assume $s_{k+1} = p-1$, then $e\Delta(s)$ is given in Proposition 4.1(c). The order on the weights of the composition factors is decreasing, so the weights of the composition factors of the uniserial module $eT(s)/e\Delta(s)$ are increasing from top to bottom. It follows that all Δ-quotients occuring in $eT(s)/e\Delta(s)$ must be simple. Since $0 \neq eL(t)$ occurs in $eT(s)/e\Delta(s)$, we have that $e\Delta(t)$ is simple. By Proposition 4.1(a) we have $t - (2 + 2\hat{s}_k)p^k < \tau$, that is $s - 2p^{k+1} < \tau$. Then also $t - (2p^i - 2)p^{k+1} < t - (2 + 2\hat{s}_k)p^k < \tau$ for $i \geqslant 1$ and it follows that $e\Delta(s)$ has only length two, and $e\Delta(t) = eL(t)$. So, as stated, we have that $eT(s)$ is uniserial of length three. $\square$

5. Applications

5.1. We will now deduce results for the Ringel duals of Schur algebras $S(2,d)$. This is possible because there are infinitely many degrees r for which $S(2,r)'$ is Morita equivalent to a Schur algebra such that the order of the weights is reversed; this was proved in [10].

To be explicit, fix an integer a with $2 \leqslant a \leqslant p$, and set $r_k = ap^k - 2$ where $k \geqslant 1$, and assume $r > p^2$. Let $r = r_k$ or $r_k - 1$, and set $\tilde{r} = r_k$ or $r_k - 1$, where this is determined uniquely by requiring $r_k - r \equiv \tilde{r} \bmod 2$. Then $S(2,r)'$ is Morita equivalent to $S{:=}S(2,\tilde{r})$ (see [10]) such that $L_{S(2,r)'}(s)$ is identified with $L_S(r_k - s)$ and $\Delta_{S(2,r)'}(s)$ with $\Delta_S(r_k - s)$, and similarly for tilting modules. (Actually, all algebras $S(2,r)$ and $S(2, r \pm 1)$ for $r = r_k$ are Ringel self-dual, but the correspondence of the labelling can be more complicated.) In this section the idempotent e is characterized by $eL_S(m) \neq 0$ if and only if $m \geqslant r_k - d$.

5.2. For arbitrary $d \geqslant 1$, we wish to identify $S(2,d)'$ with a good subalgebra (see Section 1.1) of a Schur algebra. Take $r_k > d$ and $r_k > p^2$. Let $r = r_k$ or $r_k - 1$ be such that $d \equiv r \bmod 2$. Define $\tilde{r}$ as above. The following theorem implies that the quiver and the submodule structure of standard modules and tilting modules for $S(2,d)'$ is the same as that for appropriate good subalgebras of Schur algebras, and can be read off from Section 4.

Theorem 5.1. *The algebra $S(2,d)'$ is Morita equivalent as a quasi-hereditary algebra to a good subalgebra eSe of $S = S(2,\tilde{r})$ where the weight s is identified with the weight $r_k - s$. Here e is characterized by $eL_S(m) \neq 0$ if and only if $m \geqslant r_k - d$.*

Proof. For any n with $d \leqslant n$ of the same parity, $S(2,d)'$ is a good subalgebra $e'S(2,n)'e'$ of $S(2,n)'$; to see this apply Lemma 1.1 together with [8]. Here $L_{S(2,d)'}(s)$ is identified with $e'L_{S(2,n)'}(s)$, etc. Take $n = r$ where r is as above. □

5.3. Let $G = GL_d(K)$ where d is arbitrary. In [1], Adamovich determined the submodule structure of Weyl modules $\Delta(\gamma)$ for G, where γ is a partition with at most two columns. Any such Weyl module is multiplicity-free, and a complete combinatorial description of the submodule lattice was obtained. Recall that such Weyl modules $\Delta(\gamma)$ are identified with standard modules for $S(d,d)$. Our results give a different approach to this result.

Corollary 5.1. *Let $\gamma = \lambda'$ be the conjugate of the partition $\lambda = (v,u)$ of d. Let $r_k > d$ and $r_k > p^2$, and let $S = (2,\tilde{r})$ be such that $S(2,d)'$ is a good subalgebra of S. Then $\Delta(\gamma)$ can be identified with $e\Delta_S(\tilde{s})$ for $\tilde{s} = r_k - s$ and $s = v - u$.*

The submodule structure of $\Delta(s)^{\circ} \cong \nabla(s) \cong S^s(E)$ is completely described in [7], and this gives then also a complete description of the submodule lattice of $e\Delta(s)$ and hence of $\Delta(\gamma)$.

Proof. In [5], Donkin proved that the Schur algebra $S(d,d)'$ is Morita equivalent to $S(d,d)$ as a quasi-hereditary algebra, where $L_{S(d,d)'}(\lambda)$ is identified with $L_{S(d,d)}(\lambda')$ and $\Delta_{S(d,d)'}(\lambda)$ with $\Delta_{S(d,d)}(\lambda')$. It is also known that $S(2,d)$ is a good subalgebra of $S(d,d)$, with Γ the partitions with at most two parts (see for example [9]). Hence by Ringel duality, $S(2,d)$ is a good quotient of $S(d,d)'$. It follows that the standard module for $S(2,d)'$ corresponding to partition (v,u) is identified with $\Delta_{S(d,d)}(2^u, 1^{v-u})$. Set $S = S(2,\tilde{r})$, then as in Section 5.2 we have $\Delta_{S(2,d)'}(s) \cong e\Delta(r_k - s)$. □

5.4. We consider now representations of a symmetric group $\mathscr{S}_d$, for some arbitrary degree d. Recall that I_2 is the kernel of the action of $K\mathscr{S}_d$ on the tensor space $E^{\otimes d}$ where $\dim(E) = 2$. Using Corollary 5.1, a complete description of the submodule lattice of Specht modules of $K\mathscr{S}_d$ corresponding to two-part partitions is obtained.

Corollary 5.2. *Let $d \geqslant 1$, and choose $r_k > d$ and $r, \tilde{r}$ as in Corollary* 5.1. *Then $K\mathcal{S}_d/I_2$ is Morita equivalent to $eS(2,\tilde{r})e$ where the simple module $D^{(v,u)}$ is identified with $eL(r_k - s)$, the Specht module $S^{(v,u)}$ is identified with $e\Delta(r_k - s)$ and the Young module $Y^{(v,u)}$ is identified with $eT(r_k - s)$ for $s = v - u$. If $p = 2$ and d is even, then replace e by an ef where $ef = fe$ and where f is an idempotent which annihilates precisely $L(\tilde{r})$.*

Proof. By Erdmann [9], the Ringel dual $S(2,d)'$ is Morita equivalent to $K\mathcal{S}_d/I_d$, and under this equivalence the simple module $L_{S(2,d)'}(s)$ is identified with $D^{(v,u)}$. The exception was described in Section 1.2. The corollary follows from Theorem 5.1. □

5.5. We will now translate the result in Section 3.3, and this gives the quiver of the algebra $A := K\mathcal{S}_d/I_d$; we fix d and we take $r_k > d$ and r, $\tilde{r}$ and $S = S(2,\tilde{r})$ as in Section 5.1. In addition we assume in the following that $d < p^k - 1$.

Corollary 5.3. *Let (v,u) and (g,h) be partitions of d with $u \geqslant h$. Let $s = v - u$ and suppose $s + 1 = \sum_{i \geqslant 0} s_i p^i$ p-adically, and let $t = g - h$. Then* $\mathrm{Ext}^1_A(D^{(v,u)}, D^{(g,h)}) = K$ *if either for some n, $u - h = (p - s_n)p^n$ and $s_n \neq 0$, and if $s_{n+1} = p - 1$ then $u < p^{n+1}$; or else $p = 2$ and d is even and for $(v,u) = (g,h) = (d/2 + 1, d/2 - 1)$. Otherwise* $\mathrm{Ext}^1_A(D^{(v,u)}, D^{(g,h)}) = 0$.

Proof. (1) Write $s = v - u$ and $t = g - h$, and set $\tilde{s} = r_k - s$ with p-adic expansion $\tilde{s} = \sum_{i \geqslant 0} \tilde{s}_i p^i$, and similarly $\tilde{t} = r_k - t$. By the choice of r_k we have $\tilde{s} - \tilde{t} = t - s = 2u - 2h \leqslant d < p^k$. Note that $\tilde{s} - 2p^k < r_k - d$, and moreover $\tilde{s} - 2p^{n+1} < r_k - d$ if and only if $u < p^{n+1}$. The p-adic expansion of $\tilde{s}$ is related to that of $s + 1$; namely $r_k - s = (ap^k - 1) - (s + 1)$ and $s + 1 \leqslant d + 1 \leqslant p^k - 1$, so

$$\tilde{s}_m = \begin{cases} p - 1 - s_m, & m < k, \\ a - 1, & m = k, \\ 0, & m > k. \end{cases}$$

By Corollary 5.2, the simple $D^{(v,u)}$ is identified with $eL_S(\tilde{s})$ and $D^{(g,h)}$ with $eL_S(\tilde{t})$. By Section 3.3 the extension space is non-zero if and only if $\tilde{s} - \tilde{t} = (2\tilde{s}_n + 2)p^n$, and $\tilde{s}_n \leqslant p - 2$, and moreover either $\tilde{s}_{n+1} \neq 0$ or $\tilde{s}_{n+1} = 0$ but $\tilde{s} - 2p^{n+1} < r_k - d$. Using the above statements, this can easily be seen to be equivalent to the first part of the conditions in the corollary: there exists a natural number n with $u - h = (p - s_n)p^n$ and $s_n \neq 0$, and if $s_{n+1} = p - 1$ then $u < p^{n+1}$.

(2) Next, suppose $p = 2$ and d is even; we may assume $d > 2$. Then we have $r = \tilde{r} = r_k$. Here we must replace $S(2,r)'$ by $f'S(2,r)'f'$ where f' is an idempotent in $S(2,r)'$ which annihilates $L_{S'}(0)$ (and no other simple module). We must also replace $S(2,r)$ by $fS(2,r)f$ where f is an idempotent which annihilates $L_S(r)$ (and no other simple module), and then we must take e in $fS(2,r)f$, annihilating $L_S(m)(\cong fL_S(m))$ for $\leqslant r - d$.

By Section 3.4, the quiver of $fS(2,r)f$ is obtained from the quiver of $S(2,r)$ by omitting $L(r)$ and adding a loop at the vertex corresponding to $L(r - 2)$. Translating this we get the additional self-extension as stated. □

Remark. (1) Let $A = K\mathscr{S}_d/I_d$. Suppose $p > 2$ then Ext^1_A for simple modules is the same as $\mathrm{Ext}^1_{K\mathscr{S}_d}$ for simple modules labelled by two-part partitions: When $p \geqslant 5$ then the ideal I_2 is generated by an idempotent (see [4]), and the claim follows by an elementary argument. In [16] it is proved more generally that for $n < p$

$$\mathrm{Ext}^1_{K\mathscr{S}_d}(D^\lambda, D^\mu) = \mathrm{Hom}_{K\mathscr{S}_d}(\mathrm{rad}\, S^\lambda, D^\mu), \tag{6}$$

where λ, μ are p-regular partitions with at most n parts, and λ does not strictly dominate μ. One can use this to deduce that for $p \geqslant 3$ the extensions over the factor algebra are the same as over the whole group algebra.

(2) In [16] the extensions for the case $n = 2$ are determined, for $p > 2$, by using the characterization in Eq. (6), and by using results of [1], but see the corrigenda.

(3) Now assume that $p = 2$. Some extensions over the symmetric group algebra $K\mathscr{S}_d$ for simple modules labelled by two-part partitions were given in [18]. These are usually not the same as those for the factor algebra. For example, for the factor algebra the dimension of the Ext spaces is $\leqslant 1$ (with only one exception), whereas for the group algebra, many higher dimensions occur. Our result gives a refinement, it describes the subspace of extensions which factor through the kernel of the action on the tensor space; and it shows in particular that this has a description which is uniform for arbitrary p.

5.6. In the following we exclude the partition $(r/2, r/2)$ if $p = 2$ and r is even. We will now classify uniserial Specht modules and Young modules labelled by two-part partitions. We use the notation as in Section 5.5. Let (v, u) be a partition of d, then the Specht module $S^{(v,u)}$ is identified with $e\Delta_S(\tilde{s})$ and the Young modules $Y^{(v,u)}$ is identified with $eT_S(\tilde{s})$ where $\tilde{s} = r_k - (v - u)$.

Corollary 5.4. *Let (v, u) be a partition of d and let $s = v - u$, and suppose $s + 1$ has p-adic expansion $s + 1 = \sum s_i p^i$. Let $l = \min\{j \mid s_j \neq 0\}$. Then*

(a) *$S^{(v,u)}$ is simple if and only if $u < (p - s_l)p^l$.*

(b) *$S^{(v,u)}$ is uniserial and not simple if and only if $(p - s_l)p^l \leqslant u$ and one of the following holds*:

 (i) *we have $s_{l+1} \neq 0$ and $u < p^{l+1} + (p - 1 - s_{l+b})p^{l+b}$ where $b \geqslant 1$ is minimal such that $s_{l+b} < p - 1$; in this case $S^{(v,u)}$ has length $\leqslant b + 2$;*

 (ii) *we have $s_{l+1} = 0$ and $u < (p - s_{l+c})p^{l+c}$ where $c \geqslant 1$ is minimal such that $s_{l+c} \neq 0$; in this case $S^{(v,u)}$ has length $\leqslant c + 2$.*

Remark. In case (i) the composition factors are labelled by partitions in the following (ordered) list:

$$(v, u),\ (v + p^{l+b-1}, u - p^{l+b-1}), \ldots, (v + p^{l+1}, u - p^{l+1}),$$

$$(v + (p - s_l)p^l, u - (p - s_l)p^l),$$

$$(v + (p - s_l)p^l + (p - 1 - s_{l+b})p^{l+b}, u - (p - s_l)p^l - (p - 1 - s_{l+b})p^{l+b}).$$

In case (ii) the composition factors are labelled by the partitions in the following list:

$$(v,u),\ (g,h),\ (g+(p^i-1)p^{l+1}, h-(p^i-1)p^{l+1}) \quad \text{for } 1 \leqslant i \leqslant c-1,$$
$$(g+(p^{c-1}-1)p^{l+1}+(p-1-s_{l+c})p^{l+c},$$
$$h-(p^{c-1}-1)p^{l+1}-(p-1-s_{l+c})p^{l+c}),$$

where $g=v+(p-s_l)p^l$ and $h=u-(p-s_l)p^l$.

Proof. Set $\tilde{s}=r_k-s$, so that $S^{(v,u)}$ is identified with $e\Delta_S(\tilde{s})$ (see Corollary 5.2). By Proposition 4.1 we have that $e\Delta_S(\tilde{s})$ is simple if and only if $\tilde{s}-(2+2\tilde{s}_l)p^l < r_k-d$ where $l=\min\{j \mid \tilde{s}_j \neq p-1\}$ and $\tilde{s}=\sum_{i\geqslant 0}\tilde{s}_i p^i$. As in the proof of Corollary 5.3, we have

$$\tilde{s}_i = \begin{cases} p-1-s_i, & i<k, \\ a-1, & i=k, \\ 0, & i>k. \end{cases}$$

Since $s+1 \leqslant d+1 \leqslant p^k-1$ we have $l<k$. Moreover $\tilde{s}-r_k-d=2u$. It follows that $\Delta(\tilde{s})$ is simple if and only if $u<(p-s_l)p^l$. Parts (b) and (c) follow by translating the results of Section 4.2. □

Similarly, by translating from Proposition 4.2, we obtain

Corollary 5.5. *Let (v,u) be a partition of d and let $s=v-u$, and suppose $s+1=\sum_{i\geqslant 0}s_i p^i$ p-adically. Let $l=\min\{j \mid s_j \neq 0\}$. Then the Young module $Y^{(v,u)}$ is uniserial and not simple if and only if $(p-s_l)p^l \leqslant u$ and one of the following holds*:

(i) *We have $s_{l+1}\neq 0$ and $u<(p-s_l)p^l+(p-1-s_{l+b})p^{l+b}$, where $b\geqslant 1$ is minimal such that $s_{l+b}<p-1$.*

(ii) *We have $s_{l+1}=0$ and $u<(p-1)p^{l+1}$. If so then $Y^{(v,u)}$ has length three.*

In both cases, $Y^{(v,u)}$ has two Specht quotients, labelled by (v,u) and $(v+(p-s_l)p^l, u-(p-s_l)p^l)$.

References

[1] A.M. Adamovich, Composition factors and submodules of Weyl modules, Ph.D. Thesis, Moscow State University, 1992.

[2] R. Carter, E. Cline, The submodule structure of Weyl modules for groups of type A_1, Proceedings of the Conference on Finite Groups, University of Utah, Park City, Utah, 1975. Academic Press, New York, 1976, pp. 303–311.

[3] E. Cline, Ext^1 for SL_2, Comm. Algebra 7 (1979) 107–111.

[4] C. de Concini, C. Procesi, A characteristic-free approach to invariant theory, Adv. in Math. 21 (1976) 330–354.

[5] S. Donkin, On tilting modules for algebraic groups, Math. Z. 212 (1993) 39–60.

[6] S. Donkin, The q-Schur Algebra, LMS Lecture Notes, Vol. 253, Cambridge University Press, Cambridge, 1998.

[7] S.R. Doty, The submodule structure of certain Weyl modules for groups of type A_n, J. Algebra 95 (1985) 373–383.

[8] K. Erdmann, Schur algebras of finite type, Quart. J. Math Oxford 44 (1993) 17–44.
[9] K. Erdmann, Symmetric groups and quasi-hereditary algebras, in: V. Dlab, L.L. Scott (Eds.), Finite Dimensional Algebras and Related Topics, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci, Vol. 424, Kluwer Academic Publisher, Dordrecht, 1994, pp. 123–161.
[10] K. Erdmann, A. Henke, On Ringel duality for Schur algebras, Math. Proc. Cambridge Philos. Soc., to appear.
[11] J.A. Green, Polynomial Representations of GL_n, Lecture Notes in Mathematics, Vol. 830, Springer, Berlin, 1980.
[12] A.E. Henke, Schur subalgebras and an application to the symmetric group, J. Algebra 233 (2000) 342–362.
[13] A.E. Henke, Young modules and Schur subalgebras, Ph.D. Thesis, Linacre College, University of Oxford, Trinity Term, 1999.
[14] G.D. James, The Representation Theory of the Symmetric Group, Lecture Notes in Mathematics, Vol. 682, Springer, Berlin, 1978.
[15] J.C. Jantzen, Representations of Algebraic Groups, Pure and Applied Mathematics, Vol. 131, Academic Press, Boston, 1987.
[16] A.S. Kleshchev, J. Sheth, On extensions of simple modules over symmetric and algebraic groups, J. Algebra 221 (1999) 705–722; Corrigendum: J. Algebra 238 (2001) 843–844.
[17] S. Martin, Schur Algebras and Representation Theory, Cambridge Tracts in Mathematics, Vol. 112, Cambridge University Press, Cambridge, 1993.
[18] S. Martin, L. Russell, On the ordinary quiver of the symmetric group over a field of characteristic 2, Math. Proc. Cambridge Philos. Soc. 122 (1997) 415–437.
[19] C.M. Ringel, The category of modules with good filtration over a quasi-hereditary algebra has almost split sequences, Math. Z. 208 (1991) 209–225.
[20] S. Xanthopoulos, On a question of Verma about indecomposable representations of algebraic groups and of their Lie algebras, Ph.D. Thesis, QMW College, University of London, 1992.